ON A SUPER TELESCOPING SUM REPRESENTING BINOMIAL COEFFICIENTS

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Abstract. For $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, we define the *super telescoping sum*

$$S_n(z) := \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k \geq 1}} \sum_{\substack{m_1 \geq 0 \\ m_2,\ldots,m_k \geq 1}} \prod_{i=1}^{k} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)},$$

where $M_i = m_1 + \cdots + m_i$ for $1 \leq i \leq k$, $N_0 = 0$, and $N_i = n_1 + \cdots + n_i$ for $1 \leq i \leq k$. An equivalent form of $S_n(z)$ was studied by Javad Latifi in his Ph.D. thesis. He showed, by using techniques and results from loop group theory and random matrix theory, that $S_n(z) = \binom{z+n-1}{n}$. In this note we provide a direct, elementary proof of this identity.

1. Introduction

For $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, we consider the *super telescoping sum* $S_n(z)$ defined by

$$S_n(z) := \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k \geq 1}} \sum_{\substack{m_1 \geq 0 \\ m_2,\ldots,m_k \geq 1}} \prod_{i=1}^{k} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)},$$

where $M_i = m_1 + \cdots + m_i$ for $1 \leq i \leq k$, $N_0 = 0$, and $N_i = n_1 + \cdots + n_i$ for $1 \leq i \leq k$. For $n = 1$ we have the familiar telescoping sum

$$S_1(z) = \sum_{m_1=0}^{\infty} \frac{z^2}{(z+m_1)(z+m_1+1)} = z.$$

It is not hard to see that the infinite sum over $m_1, \ldots, m_k$ is absolutely and uniformly convergent on any compact subset of $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Since $z = 0$ is a removable singularity (actually a zero) of $S_n(z)$, we know that $S_n(z)$ defines an analytic function on $\mathbb{C} \setminus \mathbb{Z}_{<0}$.

The super telescoping sum $S_n(z)$, in an equivalent form, was studied by Javad Latifi in his Ph.D. thesis. He [1] gave an indirect proof of the identity $S_n(z) = \binom{z+n-1}{n}$ based on complicated machinery from loop group theory and random matrix theory. This interesting identity arises naturally in the context of Gaussian free fields, Verblunsky sequences and loop group factorization (see [1, 2], for instance). In the same paper [1], Javad Latifi proposed the challenge of finding a direct proof of the identity for $S_n(z)$. The purpose of the present note is to give a short, elementary proof of this identity.

Theorem 1.1. *For $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ we have $S_n(z) = \binom{z+n-1}{n}$.*

Theorem 1.1 has recently been applied by Javad Latifi [1] to determine the explicit expression in terms of the Riemann zeta-function for the partition function of a lattice model with certain number-theoretic flavor.

2. Elementary Identities Involving Quotients of Falling Factorials

For any integer $n \geq 0$, the *falling factorial* $(x)_n$ is defined by

$$(x)_n := \prod_{j=0}^{n-1}(x-j) = x(x-1)\cdots(x-n+1)$$

with the obvious convention that $(x)_0 := 1$. In this section, we prove two simple identities involving quotients of falling factorials (or equivalently, quotients of binomial coefficients) which will be used in our proof of Theorem 1.1.

Lemma 2.1. *For $m \geq k \geq 1$ and $n \geq 1$, we have*

$$kn \sum_{j=k}^{m} \frac{(j-1)_{k-1}}{(n+j)_{k+1}} = \frac{(m)_k}{(n+m)_k}. \tag{1}$$

Proof. We induct on $m \geq k$. The case $m = k$ is trivial. Suppose that (1) holds for arbitrary $k, n \geq 1$ and some $m \geq k$. Then

$$\begin{aligned}
kn \sum_{j=k}^{m+1} \frac{(j-1)_{k-1}}{(n+j)_{k+1}} &= \frac{(m)_k}{(n+m)_k} + \frac{kn \cdot (m)_{k-1}}{(n+m+1)_{k+1}} \\
&= \frac{(m)_{k-1}}{(n+m+1)_{k+1}}((n+m+1)(m-k+1)+kn) \\
&= \frac{(m)_{k-1}}{(n+m+1)_{k+1}}(n+m-k+1)(m+1) \\
&= \frac{(m+1)_k}{(n+m+1)_k}.
\end{aligned}$$

Hence, (1) also holds for arbitrary $k, n \geq 1$ and $m+1$. This completes the proof. $\square$

Lemma 2.2. *For $m \geq 0$ and $n \geq 1$, we have*

$$\sum_{j=0}^{m} \frac{(m)_j}{(n+m)_{j+1}} = \frac{1}{n}. \tag{2}$$

Proof. We prove (2) by induction on $m \geq 0$. The base case $m = 0$ is clear. Suppose that (2) holds for arbitrary $n \geq 1$ and some $m \geq 0$. Then

$$\begin{aligned}
\sum_{j=0}^{m+1} \frac{(m+1)_j}{(n+m+1)_{j+1}} &= \frac{1}{n+m+1} + \frac{m+1}{n+m+1} \sum_{j=1}^{m+1} \frac{(m)_{j-1}}{(n+m)_j} \\
&= \frac{1}{n+m+1} + \frac{m+1}{n+m+1} \sum_{j=0}^{m} \frac{(m)_j}{(n+m)_{j+1}} \\
&= \frac{1}{n+m+1} + \frac{m+1}{n+m+1} \cdot \frac{1}{n} = \frac{1}{n}.
\end{aligned}$$

This shows that (2) also holds for arbitrary $n \geq 1$ and $m+1$, finishing the induction. $\square$

Remark 2.1. Although our proofs of Lemmas 2.1 and 2.2 are inductive, it may be of interest to prove these results using combinatorial arguments. For instance, we can also prove Lemma 2.2 as follows. Let $A, B \subseteq \mathbb{N}$ be two disjoint subsets of positive integers with $|A| = m$ and $|B| = n$. We consider all the permutations $(x_1, ..., x_{n+m})$ of the elements in $A \cup B$, and there are of course $(n+m)!$ of them. On the other hand, for each $1 \leq j \leq m+1$, the number of permutations $(x_1, ..., x_{n+m})$ such that $x_1, ..., x_{j-1} \in A$ and $x_j \in B$ is easily seen to be $n \cdot (m)_{j-1} \cdot (n+m-j)!$. Since for every permutation $(x_1, ..., x_{n+m})$ the smallest index j for which $x_j \in B$ satisfies $1 \leq j \leq m+1$, we have

$$\sum_{j=1}^{m+1} n \cdot (m)_{j-1} \cdot (n+m-j)! = (n+m)!\,.$$

One deduces (2) at once by dividing both sides of the above identity by $n \cdot (n+m)!$ and changing the summation index j into $j+1$.

3. $S_n(z)$ Defines A Polynomial

To prove Theorem 1.1, we introduce a slightly more general super telescoping sum. Let $m \geq 0$, $n \geq 1$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. We define the super telescoping sum $S_{m,n}(z)$ by

$$S_{m,n}(z) := \sum_{k=1}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k \geq 1}} \sum_{\substack{m_1 \geq 0 \\ m_2,\ldots,m_{k-1} \geq 1}} \prod_{i=1}^{k-1} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)} \cdot \sum_{m_k \geq 1} \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k+m)}.$$

Then $S_{m,n}(z)$ is analytic on $\mathbb{C} \setminus \mathbb{Z}_{<0}$ with $S_{m,n}(0) = 0$. In particular, $S_{0,n}(z) = S_n(z)$. For $n = 1$ one computes easily that

$$\begin{aligned} S_{m,1}(z) &= z^2 \sum_{m_1=0}^{\infty} \frac{1}{(z+m_1)(z+m_1+m+1)} \\ &= \frac{z^2}{m+1} \sum_{m_1=0}^{\infty} \left(\frac{1}{z+m_1} - \frac{1}{z+m_1+m+1} \right) \\ &= \frac{z^2}{m+1} \sum_{j=0}^{m} \frac{1}{z+j}. \end{aligned} \tag{3}$$

The following result enables us to compute $S_{m,n}(z)$ recursively for all $m \geq 0$ and $n \geq 1$.

Proposition 3.1. *For $m \geq 0$, $n \geq 2$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, we have*

$$S_{m,n}(z) = S_{m+1,n-1}(z) + \frac{z^2}{m+1} \sum_{j=1}^{m+1} \frac{S_{0,n-1}(z) - S_{j,n-1}(z)}{j}.$$

Proof. We split the sum over $n_1, ..., n_k$ in the definition of $S_{m,n}(z)$ into two parts according as $n_k = 1$ or $n_k > 1$. Explicitly, we write $S_{m,n}(z) = S'_{m,n}(z) + S''_{m,n}(z)$, where

$$S'_{m,n}(z) := \sum_{k=2}^{n} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,..,n_{k-1}\geq 1, n_k=1}} \sum_{\substack{m_1\geq 0 \\ m_2,...,m_{k-1}\geq 1}} \prod_{i=1}^{k-1} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)} \cdot \sum_{m_k\geq 1} \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k+m)},$$

$$S''_{m,n}(z) := \sum_{k=1}^{n-1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,..,n_{k-1}\geq 1, n_k>1}} \sum_{\substack{m_1\geq 0 \\ m_2,...,m_{k-1}\geq 1}} \prod_{i=1}^{k-1} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)} \cdot \sum_{m_k\geq 1} \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k+m)}.$$

Note that

$$\begin{aligned} S''_{m,n}(z) &= \sum_{k=1}^{n-1} \sum_{\substack{n_1+\cdots+(n_k-1)=n-1 \\ n_1,..,n_{k-1}\geq 1, n_k>1}} \sum_{\substack{m_1\geq 0 \\ m_2,...,m_{k-1}\geq 1}} \prod_{i=1}^{k-1} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)} \cdot \\ &\qquad \sum_{m_k\geq 1} \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k+m)} \\ &= \sum_{k=1}^{n-1} \sum_{\substack{n_1+\cdots+n_k=n-1 \\ n_1,..,n_k\geq 1}} \sum_{\substack{m_1\geq 0 \\ m_2,...,m_{k-1}\geq 1}} \prod_{i=1}^{k-1} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)} \cdot \\ &\qquad \sum_{m_k\geq 1} \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k+m+1)} = S_{m+1,n-1}(z). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} S'_{m,n}(z) &= \sum_{k=2}^{n} \sum_{\substack{n_1+\cdots+n_{k-1}=n-1 \\ n_1,..,n_{k-1}\geq 1}} \sum_{\substack{m_1\geq 0 \\ m_2,...,m_{k-1}\geq 1}} \prod_{i=1}^{k-1} \frac{z^2}{(z+M_j+N_{i-1})(z+M_i+N_i)} \cdot \\ &\qquad \sum_{m_k\geq 1} \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_{k-1}+m+1)} \\ &= \sum_{k=1}^{n-1} \sum_{\substack{n_1+\cdots+n_k=n-1 \\ n_1,..,n_k\geq 1}} \sum_{\substack{m_1\geq 0 \\ m_2,...,m_k\geq 1}} \prod_{i=1}^{k} \frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)} \cdot \\ &\qquad \sum_{m_{k+1}\geq 1} \frac{z^2}{(z+M_k+m_{k+1}+N_k)(z+M_k+m_{k+1}+N_k+m+1)}. \end{aligned}$$

The inner sum over m_{k+1} above is equal to

$$\frac{z^2}{m+1}\sum_{m_{k+1}\geq 1}\left(\frac{1}{z+M_k+m_{k+1}+N_k}-\frac{1}{z+M_k+m_{k+1}+N_k+m+1}\right)$$
$$=\frac{z^2}{m+1}\sum_{j=1}^{m+1}\frac{1}{z+M_k+N_k+j}.$$

It follows that

$$S'_{m,n}(z)=\frac{z^2}{m+1}\sum_{j=1}^{m+1}\sum_{k=1}^{n-1}\sum_{\substack{n_1+\cdots+n_k=n-1\\ n_1,..,n_k\geq 1}}\sum_{\substack{m_1\geq 0\\ m_2,\ldots,m_k\geq 1}}\prod_{i=1}^{k-1}\frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)}\cdot \frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k)(z+M_k+N_k+j)}$$
$$=\frac{z^2}{m+1}\sum_{j=1}^{m+1}\frac{1}{j}\sum_{k=1}^{n-1}\sum_{\substack{n_1+\cdots+n_k=n-1\\ n_1,..,n_k\geq 1}}\sum_{\substack{m_1\geq 0\\ m_2,\ldots,m_k\geq 1}}\prod_{i=1}^{k-1}\frac{z^2}{(z+M_i+N_{i-1})(z+M_i+N_i)}\cdot \left(\frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k)}-\frac{z^2}{(z+M_k+N_{k-1})(z+M_k+N_k+j)}\right)$$
$$=\frac{z^2}{m+1}\sum_{j=1}^{m+1}\frac{S_{0,n-1}(z)-S_{j,n-1}(z)}{j}.$$

Proposition 3.1 follows now upon combining the expressions for $S'_{m,n}(z)$ and $S''_{m,n}(z)$. ☐

Javad Latifi showed, by exploring certain cancellation patterns in the partial fraction decomposition of $S_n(z)$, that $S_n(z)$ defines a polynomial of z (see [1, Theorem 1]). It is not hard to see that this is a simple corollary of Proposition 3.1.

Corollary 3.2. *For $m\geq 0$ and $n\geq 1$, we have $(z+1)\cdots(z+m)S_{m,n}(z)\in\mathbb{Q}[z]$. In particular, we have $S_n(z)\in\mathbb{Q}[z]$.*

Proof. The case $n=1$ is easily seen to be true from (3). By Proposition 3.1 we have

$$S_{m,n+1}(z)=\left(1-\frac{z^2}{(m+1)^2}\right)S_{m+1,n}(z)+\frac{z^2}{m+1}\left(H_{m+1}S_{0,n}(z)-\sum_{j=1}^{m}\frac{S_{j,n}(z)}{j}\right),$$

where H_{m+1} is the $(m+1)$th harmonic number. The corollary follows now by induction on n based on the identity above. ☐

Remark 3.1. One can show further, by bounding the series defining $S_{m,n}(z)$ in an almost trivial way, that the polynomial $(z+1)\cdots(z+m)S_{m,n}(z)$ has degree $m+n$.

4. Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1. As we pointed out, Proposition 3.1 allows us to obtain $S_{m,n}(z)$ recursively based on (3) for $S_{m,1}(z)$. Hence to prove Theorem 1.1, we shall prove the following more general formula for $S_{m,n}(z)$.

Theorem 4.1. *For $m \geq 0$, $n \geq 1$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, we have*

$$S_{m,n}(z) = \left(1 - n\sum_{k=1}^{m} \frac{(m)_k}{(n+m)_{k+1}} \cdot \frac{k}{z+k}\right)\binom{z+n-1}{n}. \tag{4}$$

Proof. We induct on $n \geq 1$. For $n = 1$, (3) implies that

$$S_{m,1}(z) = \left(\frac{1}{m+1}\sum_{j=0}^{m} 1 - \frac{1}{m+1}\sum_{j=0}^{m}\frac{j}{z+j}\right) z = \left(1 - \frac{1}{m+1}\sum_{j=1}^{m}\frac{j}{z+j}\right) z,$$

as desired. Suppose now that (4) holds for all $m \geq 0$ and some $n \geq 1$. Then it follows from Proposition 3.1 that $S_{m,n+1}(z) = C_{m,n+1}(z)\binom{z+n}{n+1}$, where

$$C_{m,n+1}(z) := \frac{n+1}{z+n}\left(1 - n\sum_{k=1}^{m+1}\frac{(m+1)_k}{(n+m+1)_{k+1}} \cdot \frac{k}{z+k} + \frac{nz^2}{m+1}\sum_{j=1}^{m+1}\sum_{k=1}^{j}\frac{(j-1)_{k-1}}{(n+j)_{k+1}} \cdot \frac{k}{z+k}\right).$$

By Lemma 2.1 we have

$$n\sum_{j=1}^{m+1}\sum_{k=1}^{j}\frac{(j-1)_{k-1}}{(n+j)_{k+1}} \cdot \frac{k}{z+k} = n\sum_{k=1}^{m+1}\sum_{j=k}^{m+1}\frac{(j-1)_{k-1}}{(n+j)_{k+1}} \cdot \frac{k}{z+k} = \sum_{k=1}^{m+1}\frac{(m+1)_k}{(n+m+1)_k} \cdot \frac{k}{z+k},$$

which implies that

$$\frac{nz^2}{m+1}\sum_{j=1}^{m+1}\sum_{k=1}^{j}\frac{(j-1)_{k-1}}{(n+j)_{k+1}} \cdot \frac{k}{z+k} = \sum_{k=1}^{m+1}\frac{(m)_{k-1}}{(n+m+1)_k}(z-k) + \sum_{k=1}^{m+1}\frac{(m)_{k-1}}{(n+m+1)_k}\frac{k^2}{z+k}.$$

Substituting this into the definition of $C_{m,n+1}(z)$ above, we obtain

$$\begin{aligned} C_{m,n+1}(z) = \frac{n+1}{z+n}\Bigg(1 &- \sum_{k=1}^{m+1}\left(n \cdot \frac{(m+1)_k}{(n+m+1)_{k+1}} - k \cdot \frac{(m)_{k-1}}{(n+m+1)_k}\right)\frac{k}{z+k} \\ &+ \sum_{k=1}^{m+1}\frac{(m)_{k-1}}{(n+m+1)_k}(z-k)\Bigg). \end{aligned}$$

Note that

$$\begin{aligned} n \cdot \frac{(m+1)_k}{(n+m+1)_{k+1}} - k \cdot \frac{(m)_{k-1}}{(n+m+1)_k} &= \frac{(m)_{k-1}}{(n+m+1)_{k+1}}(n(m+1) - k(n+m+1-k)) \\ &= \frac{(m)_k}{(n+m+1)_{k+1}}(n-k). \end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{C_{m,n+1}(z)}{n+1} &= \frac{1}{z+n} - \sum_{k=1}^{m} \frac{(m)_k}{(n+m+1)_{k+1}} \cdot \frac{k(n-k)}{(z+k)(z+n)} + \sum_{k=1}^{m+1} \frac{(m)_{k-1}}{(n+m+1)_k} \cdot \frac{z-k}{z+n} \\ &= \left(1 + \sum_{k=1}^{m} k \cdot \frac{(m)_k}{(n+m+1)_{k+1}} - \sum_{k=1}^{m+1} (n+k) \frac{(m)_{k-1}}{(n+m+1)_k}\right) \frac{1}{z+n} \\ &\quad + \sum_{k=1}^{m+1} \frac{(m)_{k-1}}{(n+m+1)_k} - \sum_{k=1}^{m} \frac{(m)_k}{(n+m+1)_{k+1}} \cdot \frac{k}{z+k}.\end{aligned}$$

By Lemma 2.2 we see that

$$\sum_{k=1}^{m+1} \frac{(m)_{k-1}}{(n+m+1)_k} = \sum_{k=0}^{m} \frac{(m)_k}{(n+m+1)_{k+1}} = \frac{1}{n+1}$$

and that

$$\begin{aligned}\sum_{k=1}^{m+1} (n+k) \frac{(m)_{k-1}}{(n+m+1)_k} &= \sum_{k=0}^{m} (n+1+k) \frac{(m)_k}{(n+m+1)_{k+1}} \\ &= (n+1) \sum_{k=0}^{m} \frac{(m)_k}{(n+m+1)_{k+1}} + \sum_{k=1}^{m} k \cdot \frac{(m)_k}{(n+m+1)_{k+1}} \\ &= 1 + \sum_{k=1}^{m} k \cdot \frac{(m)_k}{(n+m+1)_{k+1}}.\end{aligned}$$

It follows that

$$\frac{C_{m,n+1}(z)}{n+1} = \frac{1}{n+1} - \sum_{k=1}^{m} \frac{(m)_k}{(n+m+1)_{k+1}} \cdot \frac{k}{z+k}.$$

Therefore, we conclude that

$$S_{m,n+1}(z) = C_{m,n+1}(z) \binom{z+n}{n+1} = \left(1 - (n+1) \sum_{k=1}^{m} \frac{(m)_k}{(n+m+1)_{k+1}} \cdot \frac{k}{z+k}\right) \binom{z+n}{n+1}.$$

By induction, (4) holds for all $m \geq 0$ and all $n \geq 1$. □

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